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by

Joan R. Hundhausen

A GENERATING OPERATOR FOR SOLUTIONS OF CERTAIN PARTIAL DIFFERENCE AND DIFFERENTIAL EQUATIONS

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ABSTRACT

Let M be a partial difference operator of the form $M = \sum_{m,n} a_{mn} X^m Y^n$, where a_{mn} are complex constants and X and Y are the fundamental translation operators. A related operator Z_M is shown to commute with M and may be used to generate a sequence of solutions of the equation $Mf = 0$ from a known solution. A parallel theory is developed for the partial differential operator $\mathcal{M} = \sum_{s=0}^{n} \alpha_s \frac{\partial^n}{\partial x^{n-s} \partial y^s}$. Examples are presented for both the discrete and continuous cases.

I. INTRODUCTION

This report concerns an algebraic method for generating a sequence of solutions of certain types of partial difference or differential equations from a known solution. In the discrete case, the theory is applied to complex-valued lattice functions (those functions $f(x, y)$ defined at discrete points of the complex plane) that satisfy the difference equation $Mf(x, y) = \sum_{m,n} a_{mn} X^m Y^n f(x, y) = 0$, where a_{mn} are complex constants and m and n range over a finite set of integers. The generating operator Z_M is defined in terms of certain "derived" operators associated with M.

Algebraic generating processes for harmonic and polyharmonic operators have been studied by Duffin and Shelly.[1] Other types of generating processes (as intended here) have been devised for various special forms of discrete operators; these include methods of differentiation, integration,[2] and convolution of solutions.[3,4] The process described in this report has quite general applicability in both the discrete and the continuous versions.

The approximation relationship between M and $\mathcal{M}$ is discussed in Sec. III and provides a transition between the treatments of the discrete and the continuous cases. In the continuous case, the differential equation has completely linear homogeneous form, namely,

$$\sum_{s=0}^{n} \alpha_s \frac{\partial^n f}{\partial x^{n-s} \partial y^s} = 0 \quad ,$$

with α_s complex constants. Again a generating operator $\mathcal{Z}_{\mathcal{M}}$ is introduced. Although the more orthodox approach is to derive discrete analogs from the better-known continuous theorems, this case exemplifies a statement appearing in the Editors' Foreword to the text by Miller.[5] "It is possible to derive theorems about differential equations from theorems on difference operators, and the methods might be more transparent in the latter case." Therefore, the discussion of the discrete case precedes that of the continuous case in this report.

Several applications of the generating process are presented in Sec. V. In the continuous case, the effect of $\mathcal{Z}_{\mathcal{M}}$ upon one form of the general solution of the pertinent differential equation is shown. A particularly interesting application in the discrete case lies in the context of the theory of discrete analytic functions. Here a modification of the generating operator coincides with an operator introduced by Duffin,[1] which is useful to generate a sequence of discrete analytic polynomials.

II. THE GENERATING OPERATOR IN THE CONTEXT OF DIFFERENCE EQUATIONS

A. Partial Difference Operators and Their Derived Operators

In preparation for development of the theory in the discrete case, we place a square grid of width h upon the complex plane, and designate as lattice functions $\varphi(x, y)$ those complex-valued functions defined at the points (x, y) where $x = kh$, $y = lh$, k, l integers. The equivalent expression of $\varphi(x, y)$ as a function of the single complex variable z (where $z = x + iy$) is also often convenient.

The fundamental translation operators X^n and Y^n are defined by

$$X^n\varphi(x, y) = \varphi(x + nh, y) \; ; Y^n\varphi(x, y) = \varphi(x, y + nh) ,$$

or equivalently,

$$X^n\varphi(z) = \varphi(z + nh) \; ;$$

$$Y^n\varphi(z) = \varphi(z + inh), \; n = 0, \pm1, \pm2, \ldots \; .$$

The translation operators are clearly linear and commutative, and $X^0\varphi = Y^0\varphi = I\varphi = \varphi$.

Let M represent a linear difference operator of the form

$$M = \sum_{m,n} a_{mn} X^m Y^n \; , \qquad (1)$$

where the coefficients a_{mn} are complex constants and the indices m and n range over a finite set of integers. We are concerned with the family of solutions of the homogeneous difference equation $M\varphi(x, y) = 0$.

Anticipating a form of Taylor series expansion for the operator M, we introduce the associated or derived operators

$$M_{x^r y^s} = \sum_{m,n} a_{mn} m^r n^s X^m Y^n , \; r, s = 0, 1, 2, \ldots \; .$$

Noting that $M_{x^r y^s}(1) = \sum_{m,n} a_{mn} m^r n^s$, and recalling the standard form for the Taylor series expansion of a function of two variables, we may exhibit the relationship between M and its derived operators $M_{x^r y^s}$ as follows.

$$Mf(x y) = \sum_{m,n} a_{mn} e^{h\left(m\frac{\partial}{\partial x} + n\frac{\partial}{\partial y}\right)} f(x, y)$$

$$= \sum_{m,n} a_{mn} \sum_{k=0}^{\infty} \frac{1}{k!} h^k\left(m\frac{\partial}{\partial x} + n\frac{\partial}{\partial y}\right)^k f(x, y)$$

$$= M(1) f(x, y) + hM_x(1) \frac{\partial f}{\partial x}\Big|_{(x,y)}$$

$$+ hM_y(1) \frac{\partial f}{\partial y}\Big|_{(x,y)} + \frac{1}{2} h^2 M_{x^2}(1) \frac{\partial^2 f}{\partial x^2}\Big|_{(x,y)}$$

$$+ h^2 M_{xy}(1) \frac{\partial^2 f}{\partial x \partial y}\Big|_{(x,y)}$$

$$+ \frac{1}{2} h^2 M_{y^2}(1) \frac{\partial^2 f}{\partial y^2}\Big|_{(x,y)} + \ldots \; . \qquad (2)$$

This is a corresponding expansion for the derived operators themselves.

$$M_{x^r y^s} f(x, y) = M_{x^r y^s}(1) f(x, y)$$

$$+ hM_{x^{r+1} y^s}(1) \frac{\partial f}{\partial x}\Big|_{(x,y)}$$

$$+ hM_{x^r y^{s+1}}(1) \frac{\partial f}{\partial y}\Big|_{(x,y)}$$

$$+ \frac{1}{2} M_{x^{r+2} y^s}(1) \frac{\partial^2 f}{\partial x^2}\Big|_{(x,y)}$$

$$+ h^2 M_{x^{r+1} y^{s+1}} \frac{\partial^2 f}{\partial x \partial y}\Big|_{(x,y)} \ldots \; . \qquad (3)$$

The above representations clarify the essential role played by the derived operators in the determination of the differential form which M approximates. This relationship will be discussed further in Sec. III.

B. The Generating Operator Z_M

The simple relations

$$X^m Y^n(x\varphi) = x X^m Y^n \varphi + m X^m Y^n \varphi$$

$$\text{and } X^m Y^n(y\varphi) = y X^m Y^n \varphi + n X^m Y^n \varphi$$

may be used to derive the general formula

$$M_{x^r y^s}\left(x^p y^q \varphi(x, y)\right) = \left[\sum_{i=0}^{p} \sum_{j=0}^{q} \binom{p}{i} \binom{q}{j} x^{p-i} y^{q-j} M_{x^{r+i} y^{s+j}}\right] \varphi(x, y) \quad . \tag{4}$$

Given a partial difference operator M of the form used in Eq. (1), we define a related partial difference operator Z_M in terms of certain derived operators of M as

$$Z_M \equiv y M_x - x M_y \quad .$$

The following theorem shows that the operator Z_M is useful in generating additional solutions of the difference equation $Mf = 0$ when a solution is known. Our proof is based upon the condition that the relation $Mf(x, y) = 0$ holds in a suitably extensive region of the complex plane; to simplify, we will assume that it holds in a sufficiently extensive region.

Theorem: If $Mf(x, y) = 0$ in a sufficiently extensive region of the discrete plane, then $M\left(Z_M f(x, y)\right) = 0$ also.

Proof: Using the formula of Eq. (4), we have

$$\begin{aligned} M(Z_M f) &= M(y M_x - x M_y) f \\ &= y M M_x f + M_y M_x f - x M M_y f - M_x M_y f \\ &= (y M_x - x M_y) Mf \\ &= Z_M Mf \\ &= 0 \quad . \end{aligned}$$

The latter conclusion is drawn on the assumption that $Mf = 0$ in a region containing at least each point $(x + mh, y + nh)$ where the pair (m, n) appears in the summation formula [Eq. (1)] for M.

Corollary: If $Mf(x, y) = 0$ in a sufficiently extensive region of the discrete plane, then $M\left(Z^k f(x, y)\right) = 0$, $k = 2, 3, 4, \ldots$.

The proof, again depending upon an extension of the assumption mentioned above, follows easily by induction. Indeed, this assumption is clearly sufficient in all cases, although it may not be necessary in certain special cases.

The powers of the operator Z_M may be developed with the aid of the formula in Eq. (4). For example,

$$\begin{aligned} Z_M^2 &= (yM_x - xM_y)(yM_x - xM_y) \\ &= x^2(M_y)^2 - 2xyM_xM_y + y^2(M_x)^2 + yM_{xy}M_x \\ &\quad + xM_{xy}M_y - yM_{x^2}M_y - xM_{y^2}M_x \quad . \end{aligned}$$

Likewise, the notation M^p indicates that the operator M is to be applied p times in succession; for example, $M^2 = \sum_{m,n} \sum_{k,s} a_{mn} a_{ks} X^{m+k} Y^{n+s}$, where m and k, n and s have the same ranges, respectively. The theorem above generalizes easily to the

Theorem: If $M^p f(x, y) = 0$ in a sufficiently extensive region of the discrete plane, then $M^p\left(Z_M f(x, y)\right) = 0$ also.

Finally, we display the Taylor series expansion for Z_M, wherein the role played by the derived operators of M is again emphasized.

$$\begin{aligned} Z_M f(x, y) &= \left(yM_x(1) - xM_y(1)\right) f(x, y) \\ &\quad + h\left(yM_{x^2}(1) - xM_{xy}(1)\right) \frac{\partial f}{\partial x}\bigg|_{(x,y)} \\ &\quad + h\left(yM_{xy}(1) - xM_{y^2}(1)\right) \frac{\partial f}{\partial y}\bigg|_{(x,y)} \\ &\quad + \frac{1}{2}h^2\left(yM_{x^3}(1) - xM_{x^2y}(1)\right) \frac{\partial^2 f}{\partial x^2}\bigg|_{(x,y)} \\ &\quad + h^2\left(yM_{x^2y}(1) - xM_{xy^2}(1)\right) \frac{\partial^2 f}{\partial x \partial y}\bigg|_{(x,y)} \\ &\quad + \frac{1}{2}h^2\left(yM_{xy^2}(1) - xM_{y^3}(1)\right) \frac{\partial^2 f}{\partial y^2}\bigg|_{(x,y)} \\ &\quad + \cdots \quad . \end{aligned} \tag{5}$$

III. DIFFERENCE AND DIFFERENTIAL OPERATORS

A. The Nature of the Approximation

Let $\mathcal{M}$ represent a completely linear homogeneous partial differential operator of order n; that is,

$$\mathcal{M} = \sum_{s=0}^{n} \alpha_s \frac{\partial^n f}{\partial x^{n-s} \partial y^s} \quad . \tag{6}$$

In this context, the word "homogeneous" refers to the fact that all terms contain derivatives of the same order. The expansion given in Eq. (2) illustrates the fact that a difference operator M is always an approximation to a differential operator $\mathcal{M}$ in the following sense.

$$\frac{M - M(1)I}{h^q} = \mathcal{M} + O(h) \quad ,$$

$$\text{so that} \quad \lim_{h \to 0} \frac{M - M(1)I}{h^q} = \mathcal{M} \quad . \qquad (7)$$

Here the exact value of q and the exact form of $\mathcal{M}$ are uniquely determined by the values $M_{x^r y^s}(1)$, r, s = 1, 2, ... , again emphasizing the essential role played by the derived operators of M. The uniqueness follows from the stipulation that the mesh width be the same in both directions; if the mesh length were permitted to vary as some other function of the mesh width, the differential form approximated by M would not necessarily have the homogeneous character of $\mathcal{M}$.[6]

Conversely, a given differential form $\mathcal{M}$ may always be approximated by a difference operator M, which may be accomplished in a straightforward manner by simply approximating each term of M by repeated differencing of the function and finally forming a linear combination of these results. Indeed, the great variety of difference expressions (and translations thereof) that may be used to approximate derivatives makes possible the approximation of $\mathcal{M}$ by many different forms of M.

B. Example of the Approximation

The approximation of $\mathcal{M}$ by M using the expansion of Eq. (2) has both analytic and synthetic aspects.

1. If M is given in the form of Eq. (1) or in the equivalent form of a stencil — a diagram depicting the points at which functional values are to be computed together with appropriate coefficients — the values $M_{x^r y^s}(1) = \sum_{m,n} a_{mn} m^r n^s$ may be easily computed and inserted into Eq. (2) to ascertain which differential form $\mathcal{M}$ is approximated.

2. If a form $\mathcal{M}$ and the set of points (m, n) or the set of points comprising a stencil are given, Eq. (2) may be used constructively to determine the coefficients a_{mn}. Of course, success in the latter case is not always assured and depends upon a judicious choice of the set of points (m, n). This constructive aspect is particularly well treated by Collatz.[7] Many examples depicting stencils to approximate operators of the form $\mathcal{M}$ are also presented by Hidaka.[8]

An example featuring the use of M as an approximation to the Laplacian operator $\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ will be instructive. From Eq. (2) we see that M approximates Δ [in the sense of Eq. (7)] if, and only if, $M(1) = M_x(1) = M_y(1) = M_{xy}(1) = 0$, whereas $M_{x^2}(1) = M_{y^2}(1) \neq 0$. In particular, consider the case where M is a standard five-point approximation to Δ.

$$M = D \equiv XY + X^{-1}Y^{-1} + XY^{-1} + X^{-1}Y - 4I \quad ;$$
$$D_x = XY - X^{-1}Y^{-1} + XY^{-1} - X^{-1}Y \quad ;$$
$$D_y = XY - X^{-1}Y^{-1} - XY^{-1} + X^{-1}Y \quad ;$$
$$D_{y^2} = D_{x^2} = XY + X^{-1}Y^{-1} + XY^{-1} + X^{-1}Y \quad ;$$
$$D_{xy} = XY + X^{-1}Y^{-1} - XY^{-1} - X^{-1}Y \quad .$$

Note that the conditions mentioned above are satisfied, and specifically, $D_{x^2}(1) = D_{y^2}(1) = 4$. Moreover, $Df(x, y) = h^2 \Delta f(x, y) + O(h^4)$.
Note also that

$$D(x\varphi) = x D\varphi + D_x\varphi$$

$$\text{and} \quad D_x(yf) - D_y(xf) = yD_xf - xD_yf \quad ,$$

of which the continuous analogs are

$$\Delta(x\varphi) = x\Delta\varphi + 2\frac{\partial\varphi}{\partial x}$$

$$\text{and} \quad \frac{\partial}{\partial x}(yf) - \frac{\partial}{\partial y}(xf) = y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y} \quad ,$$

respectively. Indeed, it may be readily verified from Eq. (3) that $\frac{D_x}{h}$ is truly an approximation to $\frac{\partial}{\partial x}$ as $h \to 0$ and may thus be regarded as a discrete analog of this partial derivative.

Finally, it is interesting to examine the form of Z_D using Eq. (5).

$$Z_D f = \left(yD_x - xD_y\right)f$$
$$= 4\,h\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right) + O\left(h^3\right)$$
$$\text{so that} \quad \lim_{h \to 0} \frac{Z_D}{h} = 4\left(y\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial y}\right) \quad .$$

IV. THE GENERATING OPERATOR IN THE CONTEXT OF DIFFERENTIAL EQUATIONS

From the preceding discussion about approximation, it is strongly suspected that the theory of the generating operator Z_M in the discrete case has a parallel in the continuous case. Given a partial differential operator of the form in Eq. (6), we introduce the related partial differential operator

$$\mathcal{Z}_{\mathcal{M}} \equiv \frac{1}{n}\sum_{r=0}^{n-1}\Big[y\alpha_r(n-r) - x\alpha_{r+1}(r+1)\Big]\frac{\partial^{n-1}}{\partial x^{n-1-r}\partial y^{r}} \quad .$$

Note that $\mathcal{Z}_{\mathcal{M}}$ is homogeneous but has variable coefficients. The main feature of $\mathcal{Z}_{\mathcal{M}}$ is that it commutes with $\mathcal{M}$ and is useful in generating a related sequence of additional solutions of the differential equation $\mathcal{M}f = 0$ from a known solution. The proof in the continuous case is sufficiently interesting to warrant at least the presentation of an outline in this section. In the following discussion we assume that $f \in C^{2n-1}[R]$, where R is some region of the plane.

<u>Lemma</u>:

$$\mathcal{M}\Big(\mathcal{Z}_{\mathcal{M}}\, f(x,y)\Big) = \mathcal{Z}_{\mathcal{M}}\Big(\mathcal{M}f(x,y)\Big) \quad .$$

<u>Proof</u>:

$$\mathcal{M}(\mathcal{Z}_{\mathcal{M}}f) = \sum_{s=0}^{n}\alpha_s\frac{\partial^n}{\partial x^{n-s}\partial y^s}\left\{\frac{1}{n}\sum_{r=0}^{n-1}\Big[y\alpha_r(n-r) - x\alpha_{r+1}(r+1)\Big]\frac{\partial^{n-1}f}{\partial x^{n-1-r}\partial y^r}\right\}$$

$$= \frac{1}{n}\sum_{r=0}^{n}\alpha_s\sum_{r=0}^{n-1}\left[\alpha_r(n-r)\frac{\partial^n}{\partial x^{n-s}\partial y^s}\left(y\frac{\partial^{n-1}f}{\partial x^{n-1-r}\partial y^r}\right) - \alpha_{r+1}(r+1)\frac{\partial^n}{\partial x^{n-s}\partial y^s}\left(x\frac{\partial^{n-1}f}{\partial x^{n-1-r}\partial y^r}\right)\right]$$

$$= \frac{1}{n}\sum_{r=0}^{n-1}\left\{\left(\alpha_r(n-r)y\frac{\partial^{n-1}}{\partial x^{n-1-r}\partial y^r} - \alpha_{r+1}(r+1)x\frac{\partial^{n-1}}{\partial x^{n-1-r}\partial y^r}\right)\sum_{s=0}^{n}\alpha_s\frac{\partial^n f}{\partial x^{n-s}\partial y^s}\right.$$

$$\left. + \sum_{s=0}^{n}\alpha_s\left(s\alpha_r(n-r)\frac{\partial^{2n-2}f}{\partial x^{2n-r-s-1}\partial y^{r+s-1}} - (n-s)\alpha_{r+1}(r+1)\frac{\partial^{2n-2}}{\partial x^{2n-s-r-2}\partial y^{r+s}}\right)\right\}$$

$$= \mathcal{Z}_{\mathcal{M}}(\mathcal{M}f) + \frac{1}{n}\sum_{r=0}^{n-1}\sum_{s=0}^{n}\left[\alpha_s\alpha_r s(n-r)\frac{\partial^{2n-2}f}{\partial x^{2n-r-s-1}\partial y^{r+s-1}} - \alpha_s\alpha_{r+1}(n-s)(r+1)\frac{\partial^{2n-2}f}{\partial x^{2n-s-r-2}\partial y^{r+s}}\right]$$

$$= \mathcal{Z}_{\mathcal{M}}(\mathcal{M}f) + \frac{1}{n}\left(\sum_{r=0}^{n-1}\sum_{s=0}^{n-1}\alpha_{s+1}\alpha_r(s+1)(n-r)\frac{\partial^{2n-2}f}{\partial x^{2n-r-s-2}\partial y^{r+s}}\right.$$

$$\left. - \sum_{r=0}^{n-1}\sum_{s=0}^{n-1}\alpha_s\alpha_{r+1}(n-s)(r+1)\frac{\partial^{2n-2}f}{\partial x^{2n-s-r-2}\partial y^{r+s}}\right) \quad .$$

The lengthy expression in parenthesis is easily seen to vanish.

<u>Theorem</u>: If $f(x,y) \in C^{2n-1}[R]$ and $\mathcal{M}f = 0$ in R , then $\mathcal{M}(\mathcal{Z}_{\mathcal{M}}f) = 0$ in R . The above lemma readily establishes the proof of this theorem, and the corollary follows by induction on k .

<u>Corollary</u>: If $f(x,y) \in C^{n+k(n-1)}[R]$ and $\mathcal{M}f = 0$ in R , then $\mathcal{M}(\mathcal{Z}_{\mathcal{M}}^{k} f) = 0$ in R , where k may vary over the natural integers.

The operator $\mathcal{Z}_{\mathcal{M}}^{k}$ remains linear with order $k(n-1)$, but is no longer homogeneous. The explicit form of $\mathcal{Z}_{\mathcal{M}}^{k}$ may be established with the aid of Leibnitz's rule; for example,

$$\mathcal{Z}_{\mathcal{M}}^{2} f = \frac{1}{n^2}\sum_{r=0}^{n-1}\sum_{s=0}^{n-1}\Big[y\alpha_r(n-r) - x\alpha_{r+1}(r+1)\Big]\cdot$$

$$\left\{\Big[y\alpha_s(n-s) - x\alpha_{s+1}(s+1)\Big]\cdot\right.$$

$$\frac{\partial^{2n-2}f}{\partial x^{2n-2-r-s}\partial y^{r+s}} + \frac{\partial^{2n-4}}{\partial x^{2n-3-r-s}\partial y^{r+s-1}}\cdot$$

$$\left.\left(\alpha_s(n-s)r\frac{\partial f}{\partial x} - \alpha_{s+1}(s+1)(n-1-r)\frac{\partial f}{\partial y}\right)\right\}.$$

Repeated application of the lemma yields the slightly more general

<u>Theorem</u>: If $f(x,y) \in C^{(p+1)n-1}[R]$ and $\mathcal{M}^p f = 0$ in R, then $\mathcal{M}^p(\mathcal{Z}_{\mathcal{M}} f) = 0$ in R, where p may vary over the natural integers.

V. EXAMPLES AND APPLICATIONS

A. The Continuous Case

Because the form of the general solution of the partial differential equation $\mathcal{M}f = \sum_{s=0}^{n} \alpha_s \frac{\partial^n f}{\partial x^{n-s} \partial y^s} = 0$ is known,[9] it is a straightforward matter to examine the result of applying the corresponding $\mathcal{Z}_{\mathcal{M}}$ to the general solution. Having done this, we focus attention again upon the special case $\mathcal{M} = \Delta$.

The general solution of the equation $\mathcal{M}f = 0$ is obtained by examining the roots of the auxiliary algebraic equation $P(t) \equiv \sum_{s=0}^{n} \alpha_s t^s = 0$. The form of the general solution varies accordingly as the n roots of $P(t) = 0$ are real, distinct, repeated, complex, or some combination of these. For the sake of brevity, we consider only the situation in which the roots of $P(t) = 0$, namely, $m_1, m_2, \cdots, m_n$, are real and distinct. Then the general solution is

$$f_G(x,y) = A_1(y + m_1 x) + A_2(y + m_2 x) + \cdots + A_n(y + m_n x) \quad ,$$

where the A_i are arbitrary but sufficiently differentiable functions of the variables indicated.

Now

$$\mathcal{Z}_{\mathcal{M}} f = \left(y\alpha_0 - \frac{x\alpha_1}{n}\right)\frac{\partial^{n-1} f}{\partial x^{n-1}} + \left(\frac{y\alpha_1(n-1)}{n} - \frac{x\alpha_2}{n}\right)\frac{\partial^{n-1} f}{\partial x^{n-2}\partial y} + \cdots + \left(\frac{y\alpha_{n-1}}{n} - x\alpha_n\right)\frac{\partial^{n-1} f}{\partial y^{n-1}} \quad ,$$

and application to the general solution yields, after some algebraic manipulation,

$$\mathcal{Z}_{\mathcal{M}} f_G = \frac{1}{n}\Big\{P'(m_1)(y + m_1 x)\, A_1^{(n-1)}(y + m_1 x) + P'(m_2)(y + m_2 x)\, A_2^{(n-1)}(y + m_2 x) + \cdots + P'(m_n)(y + m_n x)\, A_n^{(n-1)}(y + m_n x)\Big\} \; .$$

Consider now the special case

$$\mathcal{M}f \equiv \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 \; ; \; \mathcal{Z}_{\Delta} \equiv y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \; ;$$

$$f_G(x,y) = A(x + iy) + B(x - iy) = A(z) + B(\bar{z}) \quad .$$

In particular, $\mathcal{Z}_{\Delta} A(z) = -iz A'(z)$. An interesting result is elicited by choosing $A(z) = z^k \equiv u_k(x,y) + i v_k(x,y)$, where u_k and v_k are real harmonic polynomials. Then, because $\mathcal{Z}_{\Delta}$ is a real linear operator, from $\mathcal{Z}_{\Delta}(z^k) = -ik z^k = k v_k - ik u_k$, we may conclude that

$$\mathcal{Z}_{\Delta} u_k = k v_k \quad \text{and} \quad \mathcal{Z}_{\Delta} v_k = -k u_k \quad .$$

$\mathcal{Z}_{\Delta}$ operating upon either member of the pair u_k, v_k yields k times the harmonic conjugate of that member.

B. The Discrete Case

To illustrate the discrete case, we discuss the application of Z_M first to the simple example of Pascal's triangle, and second in the context of the theory of discrete analytic functions.

The difference equation governing the numbers in Pascal's triangle is

$$f(x+1, y+1) - f(x+1, y) - f(x,y) = 0 \quad ,$$

$$\text{or} \quad Mf \equiv (XY - X - I)f = 0 \quad .$$

A standard operator technique[10] for solution of such equations yields

$$f(x,y) = \left(\frac{1}{Y - I}\right)^x \varphi(y) \quad .$$

The initial conditions $f(x,0) = 0$ for $x \neq 0$ while $f(0,0) = 1$ determine that $\varphi(y) = f(0,y) = 1$, yielding

$$f(x,y) = (Y - I)^{-x}\, 1 \quad .$$

Here, $(Y - I)^{-1}$ is to be interpreted as indefinite summation with respect to the discrete variable y. It is the inverse operation of differencing and may be regarded as the discrete analog of indefinite integration. The standard rules of repeated indefinite summation, together with the initial conditions, yield the particular solution

$$f_p(x,y) = \binom{y}{x} \quad .$$

Finally, $Z_M = y(XY - X) - xXY$, and application of Z_M to the particular solution $\binom{y}{x}$ yields $\binom{y}{x+1}$, which may be interpreted as a horizontal translation of the solution.

An interesting application of Z_M lies in the context of discrete analytic function theory. The complex form of the Cauchy-Riemann equations is $\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0$, and a complex function f is termed analytic in the continuous theory when $\frac{\partial f}{\partial \bar{z}} = 0$. By analogy, a discrete analytic function f satisfies $Lf = 0$ in some region of the discrete plane, where L is termed a discrete analytic operator. For detailed treatment of the properties of such operators, see Duffin[1] and Hundhausen;[11] the property pertinent here is that L is a discrete approximation to $\frac{\partial}{\partial \bar{z}}$.

From the expansion of Eq. (5), we find that necessary and sufficient conditions for a discrete operator $L - L(1)I$ to simulate $\frac{\partial}{\partial \bar{z}}$ in the sense of Eq. (7) are

$$L_y(1) = i\,L_x(1) \neq 0 \quad . \tag{8}$$

If these conditions are used to characterize a family of discrete operators, it is found that the family thus characterized is identical with that for which the corresponding family of generating operators $Z_L \left(= Z_{L-L(1)I}\right)$ simulates multiplication by z. Briefly, the expansions of Eqs. (2) and (5) become

$$\left[L - L(1)I\right]f = hL_x(1)\,\frac{\partial f}{\partial \bar{z}} + \mathcal{O}(h^2) \;;$$

$$Z_L f = Z_{L-L(1)I} f = -i\,L_x(1)\,zf + \mathcal{O}(h) \quad .$$

For the family of operators satisfying Eq. (8), the notation used throughout this report becomes particularly descriptive of the analogy between the discrete and continuous cases. The theorems of Secs. II and IV may be concisely phrased, respectively, as

$$Lf = 0 \;\Rightarrow\; L(Z_L f) = 0 \quad ,$$

and

$$\frac{\partial f}{\partial \bar{z}} = 0 \;\Rightarrow\; \frac{\partial}{\partial \bar{z}}(zf) = 0 \quad .$$

Therefore, the theory presented in this report becomes useful in generating a sequence of discrete analytic functions. We conclude with an important example of the generating process as developed by Duffin[1] for the case of the discrete analytic operator $L = I + iX - XY - iY$. Duffin introduces the operator

$$Z = \frac{1}{4}\left[z(I + X + XY + Y) - i\bar{z}(I - X + XY - Y)\right]$$

and shows that if f is discrete analytic, then Zf is also discrete analytic. Algebraic simplification and use of the relation $Lf = 0$ show that Zf is a variation of $Z_L f$ as treated in this report; indeed, $Zf = \left(\frac{1}{2} - \frac{1}{2}i\right) Z_L f$, where $Z_L = y\,L_x - x\,L_y$. To achieve greater symmetry relative to the point of application, Duffin forms a new operator $\mathbf{Z}$ from the average of Z applied at the four points z, $z - 1$, $z - i$, and $z - 1 - i$, and finally establishes the interesting relation

$$\mathbf{Z}\,z^{(n)} = z^{(n+1)} \quad . \tag{9}$$

Here $z^{(n)}$ is the n^{th} member of the sequence of discrete analytic polynomials, which were originally defined by a process of recursive indefinite discrete integration with $z^{(0)} \equiv 1$; Eq. (9) provides an alternate (and simpler) method of generating this particular sequence of functions. It also may be considered a simulation of multiplication in the continuous case.

REFERENCES

1. R. J. Duffin and E. P. Shelly, "Difference Equations of Polyharmonic Type," Duke Math. J. 25, No. 2, 209-238 (1958).

2. R. J. Duffin, "Basic Properties of Discrete Analytic Functions," Duke Math. J. 23, 335-363 (1956).

3. R. J. Duffin and C. S. Duris, "A Convolution Product for Discrete Function Theory," Duke Math. J. 31, No. 2, 199-220 (1964).

4. R. J. Duffin and Joan Rohrer, "A Convolution Product for the Solutions of Partial Difference Equations," Duke Math. J. 35, No. 4, 683-698 (1968).

5. Kenneth S. Miller, Linear Difference Equations (W. A. Benjamin, Inc., New York, 1968).

6. S. K. Godunov and V. S. Ryabenki, Theory of Difference Schemes (North-Holland Publishing Co., Amsterdam, 1964).

7. L. Collatz, Numerical Treatment of Differential Equations (Springer-Verlag, Berlin, 1960), 3rd Ed.

8. K. Hidaka, "Stencils for Integrating Partial Differential Equations of Mathematical Physics," Mathematica Japonicae 2, 27-34 (1950).

9. Lester R. Ford, Differential Equations (McGraw-Hill Book Co., Inc., New York, 1955).

10. Charles Jordan, Calculus of Finite Differences (Chelsea Publishing Co., New York, 1947), 2nd Ed.

11. Joan R. Hundhausen, "A Generalization of Discrete Analytic and Harmonic Functions," J. Math. Anal. Appl. 25, No. 3, 628-652 (1969).

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